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BUCKLING OF STIFFENED PLATES UNDER AXIAL COMPRESSION AND LATERAL PRESSURE

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Abstract—The subject of this paper is the buckling behavior of a rectangular plate, with parallel thin-walled stiffeners attached to one side, subjected to a combination of axial compression and lateral pressure. The plate is modeled by the Von Kármán plate equations and the stiffeners by a nonlinear beam theory recently derived. An analytical solution is obtained for the buckling load corresponding to a torsional tripping mode of the stiffeners. This solution is compared with the experiments and theories of other researchers.

1. INTRODUCTION: STATE OF THE ART

Stiffened plates are a basic structural component of ships and submarines. These structures are designed with generous safety margins against overall collapse triggered by buckling. The object of analytical work is to determine design criteria to inhibit buckling at any stress less than yield. Surprisingly little material exists in the literature on the subject of the buckling of plates with welded stiffeners. Earlier work is summarized by Bleich (1952) and Argyris (1954). Kennard (1959) studied stiffeners that have an initial curvature. Adamchak (1979, 1982) and Kihl (1987) pointed out the importance of rotational constraint on the buckling load. Tvergaard (1973) investigated the imperfection sensitivity of stiffened panels under compression. The buckling behavior of composite stiffened panels was studied by Viswanathan *et al.* (1971) and Williams and Stein (1976). Van der Neut (1983) developed a theory for Z-stiffened panels in compression that could be solved with a pocket calculator. Codes requiring more powerful computers were developed by Wittrick and Williams (1974) and Smith (1968, 1975), based on a folded plate analysis. Bushnell (1985) also modeled the rings on cylindrical shells as plates. The ultimate strength of stiffened panels has been studied by Hughes (1983). Ostapenko (1988) calculated the tripping strength (torsional buckling load) of asymmetrical stiffeners under combined loading.

Recently, Danielson *et al.* (1990) analysed the tripping of an isolated beam. The beam was prevented from bending but was allowed to freely rotate about its base. In the present report the stiffeners are allowed to bend and twist but are constrained by the plate to which they are attached. The plate is initially rectangular in shape and has several parallel T-stiffeners spaced a distance b apart. At low values of the axial stress σ and lateral pressure p we suppose that the plate and stiffeners simply bend and compress symmetrically. Our object is to find the critical load at which the stiffened plate may buckle into an alternate mode (see Fig. 1).

We present an analysis that is based on the following simplifying assumptions:

(i) Each plate-stiffener unit of width b undergoes an identical deformation;
(ii) The plate obeys the nonlinear Von Kármán plate equations [see Timoshenko and Gere (1961)]. The stiffeners obey the nonlinear beam equations derived by Danielson and Hodges (1988);
(iii) The plate and stiffener material is elastic, linear and isotropic;
(iv) The extensional strains at the midsurface of the plate are negligible;
(v) Every line of particles in the beams normal to the plate midsurface remains normal to the deformed plate midsurface; i.e. the bases of the beams do not rotate relative to the plate;
(vi) The prebuckling displacement is small enough to be characterized by the linear theory;

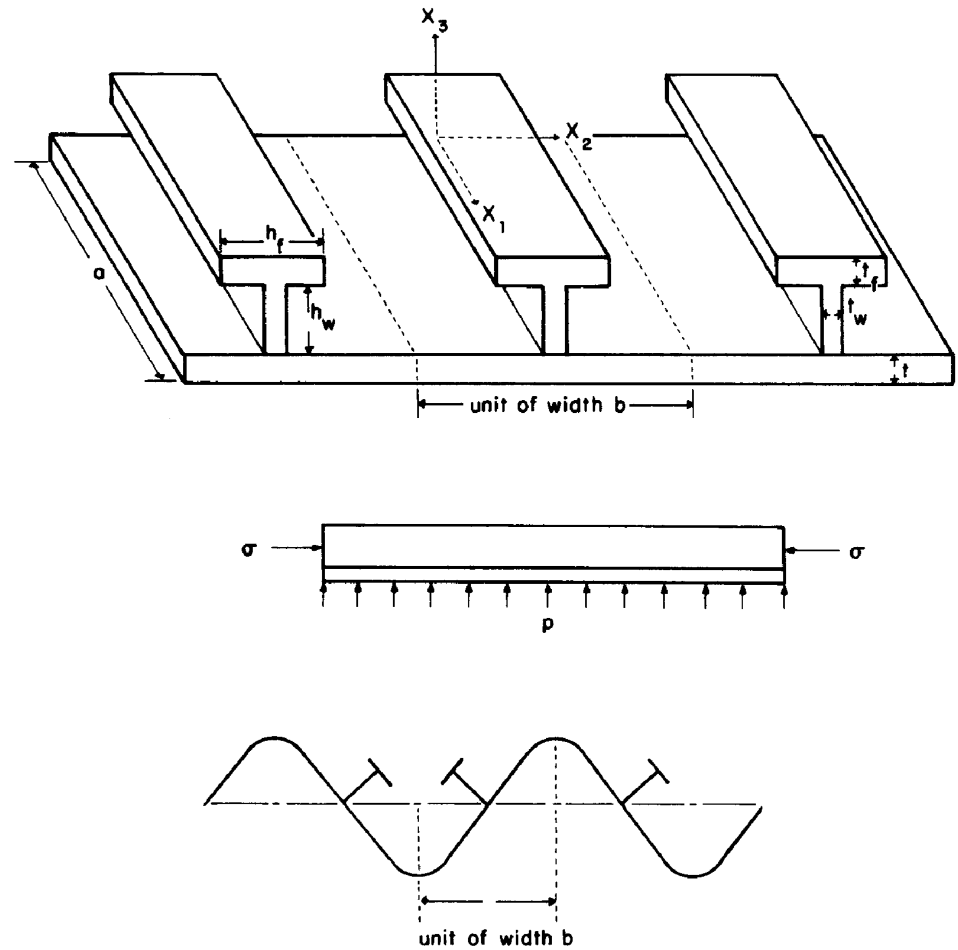

Fig. 1. Stiffened plate, loading applied, buckling mode.

(vii) The prebuckling displacement and the incremental buckling displacements may be approximated by the fundamental harmonic in their Fourier expansions;

(viii) The plate is so thin that its thickness is negligible compared to its width and length. A stiffener cross-section is so thin that its thickness is negligible compared to its height. A stiffener is so slender that its height is negligible compared to the wavelength of deformation.

2. GENERAL POTENTIAL ENERGY FUNCTIONAL

It follows from assumption (i) that we need only analyse a single plate unit containing a single stiffener. From assumptions (ii)–(iv), the potential energy of the plate plus beam is given by:

$$P[w] = \int_{-b/2}^{b/2}\int_0^a \left\{ D\left[\frac{w_{11}^2}{2} + \nu w_{11} w_{22} + \frac{w_{22}^2}{2} + (1-\nu) w_{12}^2\right] + \sigma \frac{A}{b} u_1 - pw \right\} \mathrm{d}x_1\,\mathrm{d}x_2$$
$$+ \iiint_{\text{beam}} \left(\frac{E}{2}\gamma_{11}^2 + 2G\gamma_{12}^2 + 2G\gamma_{13}^2\right) \mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3. \quad (1)$$

Here (x_1, x_2, x_3) are Cartesian coordinates measured from the midpoint of an edge of the plate unit, which has length a, width b, thickness t and cross-sectional area A. The elastic constants are defined by

$$D = \frac{Et^3}{12(1-\nu^2)}, \quad G = \frac{E}{2(1+\nu)},$$

where E is Young's modulus and ν is Poisson's ratio. The displacements of the plate midsurface in the x_1 and x_3 directions are denoted by $u(x_1, x_2)$ and $w(x_1, x_2)$, respectively. Subscripts on u or w denote partial differentiation with respect to the coordinates x_1 or x_2; i.e. $w_{12} = \partial^2 w/\partial x_1 \partial x_2$. The extensional strain at the midsurface of the plate in the x_1 direction is given by

$$e_{11} = u_1 + \tfrac{1}{2}w_1^2 \approx 0.$$

The strains in the beam are denoted by $\gamma_{11}(x_1, x_2, x_3)$, $\gamma_{12}(x_1, x_2, x_3)$ and $\gamma_{13}(x_1, x_2, x_3)$ and are related to the displacements by eqns (9)–(10) of Danielson and Hodges (1988), which upon invoking assumption (v), are transformed into:

$$\begin{aligned}
\gamma_{11} &= E_{11} + E_{12}\phi_3 - E_{13}\phi_2 + \tfrac{1}{2}\phi_2^2 + \tfrac{1}{2}\phi_3^2, \\
\gamma_{12} &= E_{12} - \tfrac{1}{2}E_{11}\phi_3, \quad \gamma_{13} = E_{13} + \tfrac{1}{2}E_{11}\phi_2, \\
E_{11} &= -x_3\bar{w}_{11} + \lambda\bar{w}_{112}, \quad E_{12} = -\tfrac{1}{2}x_3\bar{w}_{12} + \tfrac{1}{2}\lambda_2\bar{w}_{12}, \\
E_{13} &= \tfrac{1}{2}x_2\bar{w}_{12} + \tfrac{1}{2}\lambda_3\bar{w}_{12} + \tfrac{1}{2}\lambda\bar{w}_{11}\bar{w}_{12}, \\
\phi_2 &= -\tfrac{1}{2}x_2\bar{w}_{12} + \tfrac{1}{2}\lambda_3\bar{w}_{12} - \tfrac{1}{2}\lambda\bar{w}_{11}\bar{w}_{12}, \\
\phi_3 &= -\tfrac{1}{2}x_3\bar{w}_{12} - \tfrac{1}{2}\lambda_2\bar{w}_{12}.
\end{aligned}$$

Here $\lambda(x_2, x_3)$ is the Saint-Venant warping function for the beam cross-section; subscripts on λ denote partial differentiation with respect to x_2 and x_3. Bars over w denote its value at the beam axis; i.e. $\bar{w}(x_1) = w(x_1, 0)$.

Substituting these relations into (1) and neglecting higher than cubic terms in the displacements (these are not needed in our subsequent analysis), we obtain a lengthy expression for the potential energy which forms the basis for our subsequent analysis. Among all the functions satisfying the geometric or natural boundary conditions the one which causes the potential energy to be a minimum is the equilibrium state. We suppose that the outer edges of the plate are free to displace in the horizontal plane and are simply supported in the vertical direction. The geometric boundary conditions are then given by:

$$w(0, x_2) = w_{11}(0, x_2) = w(a, x_2) = w_{11}(a, x_2) = 0. \tag{2}$$

3. PREBUCKLING AND QUADRATIC ENERGY FUNCTIONALS

We base our buckling analysis on the energy criterion of elastic stability. This criterion and its application are explained by Koiter (1967), Danielson (1974) and Budiansky (1974). The prebuckling equilibrium state in the plate is denoted by $\mathring{w}$. Since the prebuckling state is assumed to be linear with vanishing midsurface strain, the prebuckling axial deflection $\mathring{u} = 0$. It follows from assumptions (ii)–(vi) that the potential energy in the prebuckling state is:

$$P[\mathring{w}] = \int_{-b/2}^{b/2}\int_0^a \left\{D\left[\frac{\mathring{w}_{11}^2}{2} + \nu\mathring{w}_{11}\mathring{w}_{22} + \frac{\mathring{w}_{22}^2}{2} + (1-\nu)\mathring{w}_{12}^2\right] - p\mathring{w}\right\} \mathrm{d}x_1\,\mathrm{d}x_2 + \frac{EI_{22}}{2}\int_0^a \mathring{\bar{w}}_{11}^2\,\mathrm{d}x_1. \tag{3}$$

Here I_{22} is the moment of inertia of the beam section about the x_2-axis:

$$I_{22} = \iint_{\substack{\text{beam} \\ \text{section}}} x_3^2 \, \mathrm{d}x_2 \, \mathrm{d}x_3.$$

The potential energy is easily recognized as being the sum of the strain energy due to bending of the plate plus the strain energy due to bending of the beam minus the work done by the pressure loading.

According to the energy criterion of elastic stability, the prebuckling equilibrium state is stable if and only if the energy functional which represents the increase of the total potential energy in a displacement field to some slightly adjacent state $(\mathring{w}+w)$ is non-negative:

$$P[\mathring{w}+w]-P[\mathring{w}] \geqslant 0. \tag{4}$$

Since the prebuckling state is an equilibrium state, the terms in (4) which are linear in the incremental displacement w must vanish. It follows that the terms $Q[w]$ in (4) which are quadratic in the incremental displacements must be non-negative:

$$Q[w] \geqslant 0.$$

The critical case of neutral equilibrium occurs when there exists a buckling mode w_{cr} satisfying

$$Q[w_{\text{cr}}] = 0, \tag{5}$$

$$Q[w \neq w_{\text{cr}}] > 0. \tag{6}$$

The eigenvalues σ_{cr} and p_{cr} which render (5) zero are the critical buckling loads.

The quadratic functional for the plate, obtained from the first integral in (1), is:

$$Q_{\text{plate}} = \int_{-b/2}^{b/2} \int_0^a \left\{ D\left[\frac{w_{11}^2}{2} + \nu w_{11} w_{22} + \frac{w_{22}^2}{2} + (1-\nu) w_{12}^2\right] - \frac{\sigma A w_1^2}{2b} \right\} \mathrm{d}x_1 \, \mathrm{d}x_2. \tag{7}$$

The quadratic functional for the beam, obtained from the last integral in (1), is:

$$Q_{\text{beam}} = \frac{EH_1}{2} \int_0^a \bar{w}_{112}^2 \, \mathrm{d}x_1 + \frac{(-EH_3+GH_2)}{2} \int_0^a \bar{\mathring{w}}_{11} \bar{w}_{12}^2 \, \mathrm{d}x_1 + \frac{GJ}{2} \int_0^a \bar{w}_{12}^2 \, \mathrm{d}x_1. \tag{8}$$

Here J is the Saint-Venant torsion constant and H_1, H_2 and H_3 are constants defined by the following integrals over the beam cross-section:

$$J = \iint_{\substack{\text{beam} \\ \text{section}}} [(x_2+\lambda_3)^2 + (x_3-\lambda_2)^2] \, \mathrm{d}x_2 \, \mathrm{d}x_3,$$

$$H_1 = \iint_{\substack{\text{beam} \\ \text{section}}} \lambda^2 \, \mathrm{d}x_2 \, \mathrm{d}x_3,$$

$$H_2 = \iint_{\substack{\text{beam} \\ \text{section}}} [x_3(x_2^2+x_3^2-\lambda_2^2-\lambda_3^2) + 2\lambda(x_2+\lambda_3)] \, \mathrm{d}x_2 \, \mathrm{d}x_3,$$

$$H_3 = \frac{1}{4} \iint_{\substack{\text{beam} \\ \text{section}}} x_3[(x_2-\lambda_3)^2 + (x_3+\lambda_2)^2 + 2(x_2^2+x_3^2-\lambda_2^2-\lambda_3^2)] \, \mathrm{d}x_2 \, \mathrm{d}x_3.$$

The total quadratic functional for the plate plus the beam is the sum of (7) and (8).

Finally, we calculate the cross-section properties for a beam composed of a thin web and a thin top flange. The web has thickness t_w and height h_w; the flange has thickness t_f and width h_f. The Saint-Venant warping function for this thin-walled cross-section is:

$$\lambda = \begin{cases} x_2(2h_w + t_f - x_3) & \text{flange,} \\ x_2 x_3 & \text{web.} \end{cases}$$

Using approximation (viii) we obtain:

$$\begin{aligned} A &= tb + t_w h_w + t_f h_f, \\ I_{22} &= \frac{t_w h_w^3}{3} + t_f h_w^2 h_f, \\ J &= \frac{t_w^3 h_w}{3} + \frac{t_f^3 h_f}{3}, \\ H_1 &= h_w^2 I_{33}, \\ H_2 &= \frac{t_f^3 h_w h_f}{3} + \frac{t_w^3 h_w^2}{6}, \\ H_3 &= \frac{t_w h_w^4}{4} + t_f h_w^3 h_f + \frac{t_f h_w h_f^3}{12}. \end{aligned}$$

Here I_{33} is the moment of inertia of the beam cross-section about the x_3 axis:

$$I_{33} = \iint_{\substack{\text{beam} \\ \text{section}}} x_2^2 \, \mathrm{d}x_2 \, \mathrm{d}x_3 = \frac{t_f h_f^3}{12}.$$

4. PREBUCKLING AND BUCKLING SOLUTIONS

In accordance with approximation (vii) and the boundary conditions (2), we represent the prebuckling displacement field by the following expression:

$$\mathring{w} = p\eta \sin\left(\frac{\pi x_1}{a}\right). \tag{9}$$

By substituting (9) into (3) and setting $\partial P/\partial \eta = 0$, we obtain the following expression for η:

$$\eta = \frac{4a^4}{\pi^5\left(D + \dfrac{EI_{22}}{b}\right)}.$$

In accordance with approximation (vii) and the boundary conditions (2), we represent the incremental buckling displacement by the following shape (an arbitrary multiplicative constant has been set equal to 1):

$$w = \sin\frac{m\pi x_1}{a} \sin\frac{\pi x_2}{b}.$$

Substitution of this buckling mode into (7) plus (8) and application of the inequality (6) leads to:

$$\sigma_{cr} \leqslant \frac{8\pi^3(2m^2-1)EH_3\eta p_{cr}}{(4m^2-1)a^2b^2A} + \frac{\pi^2 D}{bA}\left(\frac{a}{mb}+\frac{mb}{a}\right)^2 + \frac{2\pi^4 m^2 E h_w^2 I_{33}}{a^2b^2A} + \frac{2\pi^2 GJ}{b^2A}. \tag{10}$$

Here m is taken to be the integer which gives the lowest value of σ_{cr} in (10).

5. CONCLUSIONS

A simple analytical formula such as (10) that includes the effects of both the beam and lateral pressure does not appear to exist in the literature. When we set $t_w = t_f = p = 0$, formula (10) reduces to a well-known formula for the buckling load of a simply-supported plate subjected to axial compression. Equation (10) predicts that the effects of the beams and positive lateral pressure (upwards in the figure) are to increase the axial buckling load.

One of the few experiments on the buckling of stiffened plates under both axial compression and lateral pressure was performed by Smith (1975) on full scale ship grillages. In Table 1 are shown the material and geometrical parameters for two grillages tested by Smith that failed primarily by interframe tripping of longitudinal stiffeners. The grillages 1(a) and 1(b) were nominally identical, except that 1(a) was tested under compression alone, whereas 1(b) was tested under compression combined with uniform lateral pressure. The table shows the predictions of formula (10), the collapse loads measured by Smith, and the theoretical elastic buckling loads of Adamchak. Our predictions indicate that the effect of the pressure is to increase the *initial* buckling load slightly, whereas the experimental results indicate that the pressure lowers the *ultimate* collapse load slightly. The effects are to be expected: in the prebuckling state the pressure adds stiffening tension to the beams, whereas in the postbuckling state the pressure adds destabilizing deformation to the plate. Other differences between our theory and experiment can be attributed to imperfections and residual stresses as well as large deformations and plasticity in the postbuckling state. Adamchak (1979) shows how the effective width concept can be used to obtain predictions which are closer to the measured values.

In order to derive a simple formula for the buckling load we have made many approximations. Future research efforts should be directed into refining the mathematical model, modifying assumptions (i)–(viii). An improved model could include:

(i) More terms in the Fourier series expansion of the assumed solution;
(ii) Rotation of the base of the beams relative to the plate;
(iii) Geometrical imperfections and residual stresses;
(iv) Material plasticity.

Table 1. Comparison of present and previous theory with experiment.

	Parameter values
E	30,000 ksi
ν	0.3
a	48 in
b	24 in
t	0.31 in
t_w	0.28 in
h_w	5.5 in
t_f	0.56 in
h_f	3.1 in
p_{cr}	0 -Grillage 1(a); 0.015 ksi-Grillage 1(b)

	Grillage 1(a)	Grillage 1(b)
Formula (10) ($m = 1$)	45.1 ksi	45.9 ksi
Smith (1975)	27.8 ksi	27.1 ksi
Adamchak (1979)	104.7 ksi	99.4 ksi

Although experimental tests have revealed that collapse of ship grillages usually involves the tripping mode studied in this paper, other buckling modes may be involved or even dominant. In future work the parameter ranges, for which each of the various competing buckling modes are dominant, should be determined. Other possible modes include:

(i) Euler column buckling (weak stiffeners);
(ii) Local plate buckling (rigid stiffeners);
(iii) Local flange or web buckling (weak flange or web).

The use of a symbolic manipulation program is highly recommended.

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